Recent development on Noncommutative(NC) topological phases: NC AC and HMW Effects

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Outline

- **Introduction**
- **The map of QM equations from noncommutative (phase) space to commutative space.**
- **Noncommutative topological AC effect.**
- **Noncommutative HMW Effect.**
- **Noncommutative geometric phase for particles of spin-1.**
- **Conclusions.**

1. Introduction

- **Motivation:** **Why space time non-commuting? i,The NC space is necessary when one studies the low energy effective theory of D-brane with B field background, but also because in the very tiny string scale or at very high energy situation, the effects of non commutativity of both space-space and momentum-momentum may appear. ii, New physics.**

- **The purpose of this talk :** **Brief review of NCQM and present some of my recent achievements on NC topological phases (NC Aharonov and Casher (AC) effect and He-Meckellar-Wilkens(HMW) effect.**

References:

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2. The map of quantum mechanics equations from NC space to commutative space.

Define:

NC space: space-space noncommuting with momentum-momentum commuting.

NC phase space: both space-space and momentum-momentum noncommuting.

On NC space, we have

$$[\hat{x}_i, \hat{x}_j] = i\Theta_{ij}, \quad [\hat{p}_i, \hat{p}_j] = 0, \quad [\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}, \tag{1}$$

It is shown that $\hat{p}_i$, and $\hat{x}_i$ can be represented in terms of coordinates and momenta in usual quantum mechanics as

$$\hat{x}_i = x_i - \frac{1}{2\hbar}\Theta_{ij}p_j, \qquad \hat{p}_i = p_i, \tag{2}$$

On NC phase space, we have the following commutation relations:

$$[\hat{x}_i, \hat{x}_j] = i\Theta_{ij}, \quad [\hat{p}_i, \hat{p}_j] = i\bar{\Theta}_{ij}, \quad [\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}. \tag{3}$$

On NC phase space the representation in (2) become

$$\begin{cases} \hat{x}_i & = \alpha x_i - \frac{1}{2\alpha\hbar}\Theta_{ij}p_j, \\ \hat{p}_i & = \alpha p_i + \frac{1}{2\alpha\hbar}\bar{\Theta}_{ij}x_j. \end{cases} \tag{4}$$

In NC quantum mechanics, we need just to replace the usual product with the Moyal-Weyl (or star) product, then the Schr$\ddot{o}$dinger equation or Dirac equation will become the QM equations in NC quantum mechanics. Let $H(x,p)$ be the Hamiltonian operator of the usual quantum system, then the static Schr$\ddot{o}$dinger equation on NC quantum mechanics is usually written as

$$H(x,p) * \psi = E\psi, \tag{5}$$

where the Moyal-Weyl (or star) product between two functions is defined by (on NC space),

$$(f * g)(x) = e^{\frac{i}{2\hbar}\Theta_{ij}\partial_{x_i}\partial_{x_j}} f(x_i)g(x_j) = f(x)g(x) + \frac{i}{2\hbar}\Theta_{ij}\partial_i f \partial_j g|_{x_i = x_j}, \tag{6}$$

here $f(x)$ and $g(x)$ are two arbitrary functions. If we consider the case of NC phase space the definition of star product can be generalized as:

$$(f*g)(x,p) = e^{\frac{i}{2\alpha^2\hbar}\Theta_{ij}\partial_i^x\partial_j^x+\frac{i}{2\alpha^2\hbar}\bar{\Theta}_{ij}\partial_i^p\partial_j^p} f(x,p)g(x,p)$$
$$= f(x,p)g(x,p) + \frac{i}{2\alpha^2\hbar}\Theta_{ij}\partial_i^x f\partial_j^x g|_{x_i=x_j} + \frac{i}{2\alpha^2\hbar}\bar{\Theta}_{ij}\partial_i^p f\partial_j^p g|_{p_i=p_j}. \quad (7)$$

On NC space the star product can be replaced by a Bopp's shift, i.e. the star product can be changed into the ordinary product by replacing $H(x,p)$ with $H(\hat{x},\hat{p})$. Thus the Schr$\ddot{o}$dinger equation can be written as,

$$H(\hat{x}_i,\hat{p}_i)\psi = H(x_i - \frac{1}{2\hbar}\Theta_{ij}p_j, p_i)\psi = E\psi. \quad (8)$$

Here x_i and p_i are coordinate and momentum operators in usual quantum mechanics. Thus the Eq.(8) is actually defined on commutative space, and the noncommutative effects can be evaluated through the Θ related terms. Note that the Θ term always can be treated as a perturbation in QM , since $\Theta_{ij} << 1$.

To map the Schr$\ddot{o}$dinger equation from NC phase space to commutative space, the star product in Schrodinger equation should be replaced with a generalized Bopp's shift and the Schr$\ddot{o}$dinger equation then has the form

$$H(\alpha x_i - \frac{1}{2\hbar\alpha}\Theta_{ij}p_j, \alpha p_i + \frac{1}{2\hbar\alpha}\bar{\Theta}_{ij}x_j)\psi = E\psi. \quad (9)$$

For example, consider a free particle of mass m, on NC phase space, its Hamiltonian have the form,

$$\hat{H} = \frac{1}{2m}(\alpha p_i + \frac{1}{2\alpha}\bar{\Theta}_{ij}x_j)^2$$
$$= \frac{1}{2m'}(p_i + \frac{1}{2\alpha^2}\bar{\Theta}_{ij}x_j)^2, \quad (10)$$

with $m' = m/\alpha^2$.

In three dimensional NC phase space, up to the first order of Θ's, the Schr$\ddot{o}$dinger equation as,

$$i\hbar\frac{\partial\Psi(\mathbf{x},\mathbf{t})}{\partial t} = \{-\frac{\alpha^2\hbar^2}{2\mu}\nabla^2 + V(\alpha\mathbf{x}) - \frac{\mathbf{i}}{\mathbf{2\mu}}\tilde{\mathbf{x}}\cdot\nabla + \frac{\mathbf{i}}{\mathbf{2\alpha}}\nabla\mathbf{V}\cdot\tilde{\nabla}\}\mathbf{\Psi}(\mathbf{x},\mathbf{t}) \quad (11)$$

where $\tilde{\mathbf{x}}_{\mathbf{i}} = \bar{\mathbf{\Theta}}_{\mathbf{ij}}\mathbf{x}_{\mathbf{j}}$ $\tilde{\nabla} = \mathbf{\Theta}_{\mathbf{ij}}\partial_{\mathbf{j}}$. When $\alpha = 1$, which means $\bar{\Theta} = 0$, then the transformation relations from noncommutative space to commutative space become,

$$\begin{aligned} \hat{p}_i &= p_i \\ \hat{x}_i &= x_i - \tfrac{1}{2\hbar}\Theta_{ij}p_j \end{aligned} \tag{12}$$

Thus, our result reduces to the situation discussed in many papers in literature, where the Schr$\ddot{o}$dinger equation reads,

$$i\hbar\frac{\partial\Psi(\mathbf{x},\mathbf{t})}{\partial t} = [\frac{p^2}{2\mu} + V(\mathbf{x} - \frac{\mathbf{1}}{\mathbf{2\hbar}}\tilde{\mathbf{p}})]\mathbf{\Psi}(\mathbf{x},\mathbf{t}), \tag{13}$$

with $\tilde{\mathbf{p}}_{\mathbf{i}} = \mathbf{\Theta}_{\mathbf{ij}}\mathbf{p}_{\mathbf{j}}$.

3, Noncommutative AC Effect

I: Description of AC effect in 2+1 commutative space time

To begin with, let's give a brief review of AC effect in $2+1$ commutative space time. The Lagrangian for a neutral particle of spin-1/2 with an anomalous magnetic dipole moment μ_m interacting with the electromagnetic field has the form

$$L = \bar{\psi} i\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi - \frac{1}{2}\mu_m \bar{\psi}\sigma^{\mu\nu}\psi F_{\mu\nu}. \tag{14}$$

The last term in the Lagrangian is responsible for the AC effect.

We restrict the particle moves on a plane (say $x-y$ plane), then the problem can be treated in $2+1$ space time. We use the following conventions for the 2+1 dimensional metric $g_{\mu\nu}$ and the anti-symmetric tensor $\epsilon_{\mu\nu\alpha}$:

$$g_{\mu\nu} = \mathbf{diag}(1,-1,-1) \qquad \textbf{and} \qquad \epsilon_{012} = +1. \tag{15}$$

Other than to use 2×2 matrices satisfying the 2+1 dimensional Dirac algebra, we will use three 4×4 Dirac matrices which can describe spin up and down in the notional z direction for a particle and for its anti-particle. In 2+1 dimensions these Dirac matrices satisfy the following relation:

$$\gamma^\mu\gamma^\nu = g^{\mu\nu} - i\gamma^0\sigma^{12}\epsilon^{\mu\nu\lambda}\gamma_\lambda. \tag{16}$$

A particular representation is

$$\gamma^0 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} i\sigma_1 & 0 \\ 0 & i\sigma_1 \end{pmatrix}, \tag{17}$$

Then the interaction term in the Lagrangian can be written as

$$\bar{\psi}\sigma^{\mu\nu}\psi F_{\mu\nu} = F^{\mu\nu}\gamma^0\sigma^{12}\epsilon_{\mu\nu\lambda}\bar{\psi}\gamma^\lambda\psi, \tag{18}$$

with

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 \\ E^1 & 0 & -B^3 \\ E^2 & B^3 & 0 \end{pmatrix}, \tag{19}$$

where E^i and B^i are the electric and magnetic fields, respectively. The indices "1" and "2" indicate the coordinates on the $x-y$ plane along the x and y directions. The index "3" indicates the z direction. The Lagrangian now can be written as

$$L = \bar{\psi} i\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi - (1/2)\gamma^0\sigma^{12}\mu_m \epsilon_{\mu\alpha\beta}F^{\alpha\beta}\bar{\psi}\gamma^\mu\psi. \tag{20}$$

By using E-L equation, the Dirac equation of motion for a spin half neutral particle with a magnetic dipole moment μ_m is

$$(i\gamma_\mu\partial^\mu - (1/2)\gamma^0\sigma^{12}\mu_m\epsilon_{\mu\alpha\beta}F^{\alpha\beta}\gamma^\mu - m)\psi = 0, \tag{21}$$

and the solution will have the form

$$\psi = e^{-\frac{i}{2}\gamma^0\sigma^{12}\mu_m \int^x \epsilon_{\mu\alpha\beta}F^{\alpha\beta}dx^\mu}\psi_0, \tag{22}$$

where ψ_0 is the solution for electromagnetic field free case . The phase in Eq.(22) is called AC phase , we write it as

$$\phi_{AC} = -\frac{1}{2}\gamma^0\sigma^{12}\mu_m \int^x \varepsilon_{\mu\alpha\beta}F^{\alpha\beta}dx^\mu. \tag{23}$$

The AC phase above is the general AB phase for a spin-1/2 neutral particle passing through an electromagnetic field. If we consider a situation of the standard AC configuration i.e. the particle moves on a plane under the influence of an pure electric field produced by a uniformly charged infinitely long filament perpendicular to the plane, then we have

$$\phi_{AC} = -\gamma^0\sigma^{12}\mu_m \int^x \varepsilon_{0ij}F^{0i}dx^j = \gamma^0\sigma^{12}\mu_m \int^x (\hat{\vec{k}}\times\vec{E})\cdot d\vec{x} = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix} \int^x (\vec{\mu}_m \times \vec{E})\cdot d\vec{x}, \tag{24}$$

where $\hat{\vec{k}}$ is the unit vector in z direction and we assume that the magnetic dipole moment is always along this direction, i.e. $\vec{\mu}_m = \mu_m\hat{\vec{k}}$.

II: AC effect in 2+1 NC space

Just like the static Schrödinger equation on NC space[?], the Dirac equation (21) for a spin half neutral particle with a magnetic dipole moment μ_m, on NC space, can be written as

$$(i\gamma_\mu\partial^\mu - (1/2)\gamma^0\sigma^{12}\mu_m\epsilon_{\mu\alpha\beta}F^{\alpha\beta}\gamma^\mu - m) * \psi = 0, \tag{25}$$

i.e, simply replace usual product with a star product (Moyal-Weyl product), the Dirac equation in usual commuting space will change into the Dirac equation on NC space. Some features of AC effect on noncommutative space has been studied in [?] by using the star calculation, but it is still meaningful to study it again by using the method gave in reference [?], i.e. through a generalized Bopp′s shift, and the method can be easily generalized to NC phase space which will be discussed in the next section.

On NC space the star product can be replaced by a Bopp's shift, i.e the star product can be changed into ordinary product by shifting coordinates x_μ with

$$\hat{x}_\mu = x_\mu - \frac{1}{2}\Theta_{\mu\nu}p^\nu. \tag{26}$$

Now, let us consider the noncommutative Dirac equation (25), to replace the star product with ordinary product, equivalent to the Bopp's shift, the $F_{\mu\nu}$ must, up to the first order of the NC parameter Θ, be shifted by

$$F_{\mu\nu} \rightarrow \hat{F}_{\mu\nu} = F_{\mu\nu} + \frac{1}{2}\Theta^{\alpha\beta}p_\alpha\partial_\beta F_{\mu\nu}. \tag{27}$$

Then the Dirac equation for AC problem on NC space has the form

$$(i\gamma_\mu\partial^\mu - (1/2)\gamma^0\sigma^{12}\mu_m\epsilon_{\mu\alpha\beta}\hat{F}^{\alpha\beta}\gamma^\mu - m)\psi = 0. \tag{28}$$

So, on NC space, the AC phase has the form:

$$\begin{aligned} \hat{\phi}_{AC} &= -\frac{1}{2}\gamma^0\sigma^{12}\mu_m\int^x \varepsilon_{\mu\alpha\beta}\hat{F}^{\alpha\beta}dx^\mu \\ &= -\frac{1}{2}\gamma^0\sigma^{12}\mu_m\int^x \varepsilon_{\mu\alpha\beta}F^{\alpha\beta}dx^\mu - \frac{1}{4}\gamma^0\sigma^{12}\mu_m\int^x \epsilon_{\mu\alpha\beta}\Theta^{\sigma\tau}p_\sigma\partial_\tau F^{\alpha\beta}dx^\mu. \end{aligned} \tag{29}$$

This is the general AC phase for a spin-1/2 neutral particle moving in a general electromagnetic field.

In the standard AC configuration, the momentum on NC space can be written as

$$p_l = mv_l + (\vec{E} \times \vec{\mu})_l + \mathcal{O}(\theta), \tag{30}$$

where $\vec{\mu} = \mu_m \vec{\sigma}$, insert equation (100) to (123) and notice that

$$F^{\alpha\beta} \longrightarrow F^{0i} \quad \text{and} \quad \Theta^{ij} = \theta\epsilon^{ij}, \;\; \Theta^{0\mu} = \Theta^{\mu 0} = 0, \tag{31}$$

we have

$$\hat{\phi}_{AC} = \phi_{AC} + \delta\phi_{NCS}, \tag{32}$$

where ϕ_{AC} is the AC phase on commuting space given by (24), the additional phase $\delta\phi_{NCS}$, related to the non-commutativity of space, is given by

$$\begin{aligned} \delta\phi_{NCS} &= -\frac{1}{2}\gamma^0\sigma^{12}\mu_m \int^x \epsilon_{\mu 0 i}\theta\epsilon^{\alpha\beta}[mv_\alpha + (\vec{E} \times \vec{\mu})_\alpha]\partial_\beta F^{0i} dx^\mu \\ &= \frac{1}{2}\gamma^0\sigma^{12}\mu_m\theta\epsilon^{ij} \int^x [k_j + (\vec{E} \times \vec{\mu})_j](\partial_i E^2 dx^1 - \partial_i E^1 dx^2), \end{aligned} \tag{33}$$

where $k_j = mv_j$. If the spin of the neutral particle along the z direction, namely, we can chose $\vec{\mu} = \sigma_3 \mu_m \hat{\vec{k}}$, then our result here will exactly the same as the result given by B.Mirza and M.Zarei (EPJC), where the tedious star product calculation has been used.

III: AC effect in 2+1 NC phase space

The Dirac equation (21) on NC phase space can also be written as:

$$(-\gamma_\mu p^\mu - (1/2)\gamma^0\sigma^{12}\mu_m\epsilon_{\mu\alpha\beta}F^{\alpha\beta}\gamma^\mu - m) * \psi = 0, \tag{34}$$

The star product in Dirac equation on NC phase space can be placed by the usual product from the two steps, first we need to replace x_i and p_i with a generalized Bopp's shift as

$$\begin{aligned} x_\mu &\rightarrow \alpha x_i - \frac{1}{2\alpha}\Theta_{\mu\nu}p_\nu, \\ p_\mu &\rightarrow \alpha p_\mu + \frac{1}{2\alpha}\bar{\Theta}_{\mu\nu}x_\nu, \end{aligned} \tag{35}$$

and also need the partner of shift in Eq.(92) in NC phase space as,

$$F_{\mu\nu} \rightarrow \hat{F}_{\mu\nu} = \alpha F_{\mu\nu} + \frac{1}{2\alpha}\Theta^{\alpha\beta} p_\alpha \partial_\beta F_{\mu\nu}. \tag{36}$$

The Dirac equation (34) then read

$$\{-\alpha\gamma^\mu p_\mu - \frac{1}{2\alpha}\gamma^\mu \bar{\Theta}_{\mu\nu} x_\nu - (1/2)\gamma^0\sigma^{12}\mu_m \epsilon_{\mu\alpha\beta}[\alpha F^{\alpha\beta} + \frac{1}{2\alpha}\Theta^{\tau\sigma} p_\tau \partial_\sigma F^{\alpha\beta}]\gamma^\mu - m\}\psi = 0. \tag{37}$$

Because $\alpha \neq 0$, so the above Dirac equation can be recast to

$$\{-\gamma^\mu p_\mu - \frac{1}{2\alpha^2}\gamma^\mu \bar{\Theta}_{\mu\nu} x_\nu - (1/2)\gamma^0\sigma^{12}\mu_m \epsilon_{\mu\alpha\beta}[F^{\alpha\beta} + \frac{1}{2\alpha^2}\Theta^{\tau\sigma} p_\tau \partial_\sigma F^{\alpha\beta}]\gamma^\mu - m'\}\psi = 0, \tag{38}$$

where $m' = m/\alpha$. The solution to (38) is

$$\psi = e^{i\hat{\varphi}_{AC}}\psi_0, \tag{39}$$

where ψ_0 is the solution of Dirac equation for free particle with mass m', and the $\hat{\varphi}_{AC}$ stands AC phase in NC phase space, and it has the form below,

$$\begin{aligned} \hat{\varphi}_{AC} = & -\frac{1}{2}\gamma^0\sigma^{12}\mu_m \int^x \varepsilon_{\mu\alpha\beta} F^{\alpha\beta} dx^\mu \\ & -\frac{1}{2\alpha^2}\int^x \bar{\Theta}_{ij} x_j dx_i - \frac{1}{4\alpha^2}\gamma^0\sigma^{12}\mu_m \int^x \epsilon_{\mu\alpha\beta}\Theta^{\sigma\tau} p_\sigma \partial_\tau F^{\alpha\beta} dx^\mu. \end{aligned} \tag{40}$$

Equation (40) is the general AC phase in noncommutative phase space. For the standard AC case i.e. particle moves in a pure static electric field, then the AC phase reduces to

$$\begin{aligned} \hat{\varphi}_{AC} &= \phi_{AC} - \frac{1}{2\alpha^2}\int^x \bar{\Theta}_{ij} x_j dx_i - \frac{1}{2\alpha^2}\gamma^0\sigma^{12}\mu_m \int^x \epsilon_{\mu 0 i}\theta\epsilon^{\alpha\beta}[m' v_\alpha + (\vec{E}\times\vec{\mu})_\alpha]\partial_\beta F^{0i} dx^\mu \\ &= \phi_{AC} - \frac{1}{2\alpha^2}\int^x \bar{\Theta}_{ij} x_j dx_i + \frac{1}{2\alpha^2}\gamma^0\sigma^{12}\mu_m \theta\epsilon^{ij}\int^x [k'_j + (\vec{E}\times\vec{\mu})_j](\partial_i E^2 dx^1 - \partial_i E^1 dx^2), \end{aligned}$$

where $k'_j = m' v_j,\ p_l = m' v_l + (\vec{E}\times\vec{\mu})_l + \mathcal{O}(\theta)$ has been applied and we omit the second order terms in θ. Equation (41) can also be written as

$$\hat{\varphi}_{AC} = \phi_{AC} + \delta\phi_{NCS} + \delta\phi_{NCPS}, \tag{42}$$

where ϕ_{AC} is the AC phase on commuting space (see Eq. 24), $\delta\phi_{NCS}$ is the space-space non-commuting contribution to the AC phase on NC space (see Eq. 102), and the last term $\delta\phi_{NCPS}$ is given by

$$\delta\phi_{NCPS} = -\frac{1}{2\alpha^2}\int^x \bar{\Theta}_{ij}x_j dx_i + \frac{1-\alpha^3}{2\alpha^3}\gamma^0\sigma^{12}\mu_m\theta\epsilon^{ij}\int^x k_j(\partial_i E^2 dx^1 - \partial_i E^1 dx^2) + \frac{1-\alpha^2}{2\alpha^2}\gamma^0\sigma^{12}\mu_m\theta\epsilon^{ij}\int^x (\vec{E}\times\vec{\mu})_j(\partial_i E^2 dx^1 - \partial_i E^1 dx^2), \quad (43)$$

which represents the non-commutativity of the momenta. The first term in $\delta\phi_{NCPS}$ is a contribution purely from the non-commutativity of the momenta, the second term is a velocity dependent correction and the third term is a correction to the vortex of magnetic field. In 2 dimensional non-commutative plane, $\bar{\Theta}_{ij} = \bar{\theta}\epsilon_{ij}$, and the two NC parameters θ and $\bar{\theta}$ are related by $\bar{\theta} = 4\alpha^2(1-\alpha^2)/\theta$ [?]. When $\alpha = 1$, which will lead to $\bar{\theta}_{ij} = 0$, , then the AC phase on NC phase space will return to the AC phase on NC space, i.e. $\delta\phi_{NCPS} = 0$ and equations (40) and (41) will change into equations (123) and (101) respectively.

4, Noncommutative HMW Effect

I: Review of the HMW effect on 2+1 dimensional commutative space time

The HMW effect corresponds to a topological phase related to a neutral spin-1/2 particle with non-zero electric dipole moving in the magnetic field. The HMW effect was firstly discussed in 1993 by He and Meckellar and a year later, independently by Wilkens. In 1998, Dowling, Willianms and Franson point out that the HMW effect can be partially tested using metastable hydrogen atoms. Just as the AB AC effect, the HMW effect has the same importance in the literature, and the study of the correction of the space (and momenta) non-commutativity to the HMW effect will be meaningful.

In order to study the NC properties of HMW effect, a brief review of the effect in $2+1$ dimensional commutative space time is necessary. The Lagrange of a spin-1/2 neutral particle with electric dipole μ_e moving in the electromagnetic field is given by

$$L = \bar{\psi} i\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi - i\frac{1}{2}\mu_e \bar{\psi}\sigma^{\mu\nu}\gamma_5 \psi F_{\mu\nu}. \tag{44}$$

The last term in the Lagrangian represents the HMW effect. Using the identity $\sigma^{\mu\nu}\gamma_5 = (i/2)\epsilon^{\mu\nu\alpha\beta}\sigma_{\alpha\beta}$, the Lagrangian becomes

$$L = \bar{\psi} i\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi + \frac{1}{2}\mu_e \tilde{F}_{\mu\nu}\bar{\psi}\sigma^{\mu\nu}\psi, \tag{45}$$

where $\tilde{F}$ is the Hodge star of F, i.e. $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$. Similar as AB, AC other topological effects, the HMW effect is also usually studied in $2+1$ dimension, because the particle movies in a plane.

Same as the case in Noncommutative AC effect, we restrict the particle moves on a plane (say $x-y$ plane), then the problem can be treated in $2+1$ space time, and use three 4×4 Dirac matrices, which satisfy the following relation

$$\gamma^\mu\gamma^\nu = g^{\mu\nu} - i\gamma^0\sigma^{12}\epsilon^{\mu\nu\lambda}\gamma_\lambda. \tag{46}$$

A particular representation is

$$\gamma^0 \;=\; I \otimes \sigma_3, \;\; \gamma^1 = i\mathbf{diag}(1,-1) \otimes \sigma_2, \;\; \gamma^2 = iI \otimes \sigma_1. \tag{47}$$

We define

$$\mathbf{a} = -i\gamma^0\gamma^1\gamma^2 = -\gamma^0\sigma^{12} = \mathbf{diag}(1,-1) \otimes \sigma_3, \tag{48}$$

then the Lagrangian in $2+1$ dimension can further be written as

$$L = \bar{\psi} i\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi - (1/2)\mathbf{a}\mu_e \epsilon_{\alpha\beta\mu}\tilde{F}^{\alpha\beta}\bar{\psi}\gamma^\mu\psi. \tag{49}$$

By using Euler-Lagrange equation, the Dirac equation of motion for a spin half neutral particle with a electric dipole moment μ_e is

$$(i\gamma_\mu\partial^\mu + S_\mu\gamma^\mu - m)\psi = 0, \tag{50}$$

where

$$S_\mu = -(1/2)\mathbf{a}\mu_e\epsilon_{\alpha\beta\mu}\tilde{F}^{\alpha\beta}. \tag{51}$$

The solution to the Dirac equation have the form

$$\psi = e^{i\phi_{HMW}}\psi_0, \tag{52}$$

where ψ_0 is the solution for electromagnetic field free case . The phase in Eq.(24) is called HMW phase , and it has the form

$$\phi_{HMW} = \int^x S_\mu dx^\mu = -\frac{1}{2}\mathbf{a}\mu_e \int^x \varepsilon_{\alpha\beta\mu}\tilde{F}^{\alpha\beta}dx^\mu. \tag{53}$$

The HMW phase above is the general HMW phase for a spin-1/2 neutral particle passing through an electromagnetic field. When the neutral particle moves through a pure static magnetic field, $\tilde{F}^{\mu\nu}$ reduced to $\tilde{F}^{0i}$, then we have

$$\phi_{HMW} = -\mathbf{a}\mu_e \int^x \varepsilon_{0ij}\tilde{F}^{0i}dx^j = -\mathbf{a}\mu_e \int^x (\hat{k} \times \vec{B}) \cdot d\vec{x}, \tag{54}$$

where $\hat{k}$ is the unit vector in z direction and we assume that the magnetic dipole moment is always along this direction.

II: The HMW phase in noncommutative quantum mechanics

Now we are in the position to discuss the HMW topological phase in NC quantum mechanics. First let's consider the NC space case, the Dirac equation for HMW effect becomes

$$[i\gamma_\mu \partial^\mu + S_\mu \gamma^\mu - m] \star \psi = 0, \tag{55}$$

i.e. just replace normal product to a star product, then the Dirac equation in commuting space will change into the Dirac equation in NC space. We replace the star product in Dirac equation with usual product by shift NC coordinates with a Bopp's shift

$$\hat{x}_i = x_i - \frac{1}{2}\theta_{ij} p_j, \quad \hat{p}_i = p_i, \tag{56}$$

as well as a shift for the vector for vector S_μ,

$$S_\mu \to \hat{S}_\mu = S_\mu + \frac{1}{2}\theta^{\alpha\beta} p_\alpha \partial_\beta S_\mu. \tag{57}$$

Then the Dirac equation can be solved in the commuting space and the non-commutative properties can be realized through the terms related to θ. After the shift defined in Eq.(57), the Dirac equation becomes:

$$(i\gamma_\mu \partial^\mu - (1/2)\mathbf{a}\mu_e \epsilon_{\mu\alpha\beta}(\tilde{F}^{\alpha\beta} + \frac{1}{2}\theta^{\tau\sigma} p_\tau \partial_\sigma \tilde{F}^{\alpha\beta})\gamma^\mu - m)\psi = 0. \tag{58}$$

This equation is defined in commuting space and the coordinate non-commutative effect appears in θ related terms. It is easy to check that the solution to this Dirac equation has the form

$$\psi = e^{i\hat{\phi}_{HMW}} \psi_0, \tag{59}$$

where ψ_0 is the solution for electromagnetic field free case, and $\hat{\phi}_{HMW}$ is the HMW phase in NC space, which is read

$$\begin{aligned} \hat{\phi}_{HMW} &= -\frac{1}{2}\mathbf{a}\mu_e \int^x \varepsilon_{\mu\alpha\beta} \hat{\tilde{F}}^{\alpha\beta} dx^\mu \\ &= -\frac{1}{2}\mathbf{a}\mu_e \int^x \varepsilon_{\mu\alpha\beta} \tilde{F}^{\alpha\beta} dx^\mu - \frac{1}{4}\mathbf{a}\mu_e \int^x \epsilon_{\mu\alpha\beta}\theta^{\sigma\tau} p_\sigma \partial_\tau \tilde{F}^{\alpha\beta} dx^\mu. \end{aligned} \tag{60}$$

This is the general HMW phase for a spin-1/2 neutral particle moving in a general electromagnetic field.

Now let's consider the situation where only static electric field exist. Just like the case discussed in [?], the Hamiltonian of the particle in commuting space has the form:

$$H = \frac{1}{2m}\vec{\sigma}\cdot(\vec{p}+i\mu_e\vec{B})\vec{\sigma}\cdot(\vec{p}-i\mu_e\vec{B}). \tag{61}$$

By using $\vec{\nabla}\cdot\vec{B}=0$, the Eq.(123) can be recast as

$$H = \frac{1}{2m}(\vec{p}-\vec{\mu}\times\vec{B})^2 - \frac{\mu^2 B^2}{2m}, \tag{62}$$

where $\vec{\mu}=\mu_e\vec{\sigma}$, then the velocity operator can be gotten

$$v_l = \frac{\partial H}{\partial p_l} = \frac{1}{m}[p_l - (\vec{\mu}\times\vec{B})_l]. \tag{63}$$

From this equation we know that in non-commutative space, we have

$$p_l = mv_l + (\vec{\mu}\times\vec{B})_l + \mathcal{O}(\theta). \tag{64}$$

Insert equation (??) to (??) and notice that

$$\tilde{F}^{\alpha\beta} \longrightarrow \tilde{F}^{0i} \quad \text{and} \quad \theta^{ij}=\theta\epsilon^{ij},\ \theta^{0\mu}=\theta^{\mu 0}=0, \tag{65}$$

we have

$$\hat{\phi}_{HMW} = \phi_{HMW} + \delta\phi_{NCS} \tag{66}$$

where ϕ_{HMW} is the HMW phase in commuting space given by (91), the added phase $\delta\phi_{NCS}$, related to the non-commutativity of space, is given by

$$\begin{aligned}\delta\phi_{NCS} &= -\frac{1}{2}\mathbf{a}\mu_e\int^x \epsilon_{\mu 0i}\theta\epsilon^{\alpha\beta}[mv_\alpha + (\vec{\mu}\times\vec{B})_\alpha]\partial_\beta\tilde{F}^{0i}dx^\mu \\ &= \frac{1}{2}\mathbf{a}\mu_e\theta\epsilon^{ij}\int^x [k_j + (\vec{\mu}\times\vec{B})_j](\partial_i B^2 dx^1 - \partial_i B^1 dx^2),\end{aligned} \tag{67}$$

where $k_j = mv_j$, and the result here coincides with the result given in reference [?], where the tedious star product calculation has been used.

When both space-space and momentum-momentum non-commutating are considered, i.e. we study the problem on NC phase space, the Dirac

equation for the HMW model is the same as the case on NC space, but the star product and the shifts are defined in Eqs.(??) and (??). After a similar procedure as in NC space, we got the Dirac equation on NC phase space as:

$$\{-\gamma^\mu p_\mu - \frac{1}{2\alpha^2}\gamma^\mu \bar{\theta}_{\mu\nu} x_\nu - (1/2)\mathbf{a}\mu_e \epsilon_{\mu\alpha\beta}[\tilde{F}_{\alpha\beta} + \frac{1}{2\alpha^2}\theta^{\tau\sigma} p_\tau \partial_\sigma \tilde{F}_{\alpha\beta}]\gamma^\mu - m'\}\psi = 0, \quad (68)$$

where $m' = m/\alpha$. The solution to (68) is

$$\psi = e^{i\hat{\varphi}_{HMW}}\psi_0, \quad (69)$$

where ψ_0 is the solution of Dirac equation for free particle with mass m', and $\hat{\varphi}_{HMW}$ stands HMW phase in NC phase space, and it has the form below,

$$\begin{aligned}\hat{\varphi}_{HMW} &= -\frac{1}{2}\mathbf{a}\mu_e \int^x \varepsilon_{\mu\alpha\beta}\tilde{F}^{\alpha\beta} dx^\mu \\ &\quad -\frac{1}{2\alpha^2}\int^x \bar{\theta}_{ij} x_j dx_i - \frac{1}{4\alpha^2}\mathbf{a}\mu_e \int^x \epsilon_{\mu\alpha\beta}\theta^{\sigma\tau} p_\sigma \partial_\tau \tilde{F}^{\alpha\beta} dx^\mu. \end{aligned} \quad (70)$$

Equation (70)is the general HMW phase in noncommutative phase space. Once again for the case only static magnetic field exist, then the HMW phase reduces to

$$\hat{\varphi}_{HMW} = \phi_{HMW} + \delta\phi_{NCPS}, \quad (71)$$

where

$$\begin{aligned}\delta\phi_{NCPS} &= -\frac{1}{2\alpha^2}\int^x \bar{\theta}_{ij} x_j dx_i - \frac{1}{2\alpha^2}\mathbf{a}\mu_e \int^x \epsilon_{\mu 0 i}\theta\epsilon^{\alpha\beta}[m' v_\alpha + (\vec{\mu}\times\vec{B})_\alpha]\partial_\beta \tilde{F}^{0i} dx^\mu \\ &= -\frac{1}{2\alpha^2}\int^x \bar{\theta}_{ij} x_j dx_i + \frac{1}{2\alpha^2}\mathbf{a}\mu_e \theta\epsilon^{ij}\int^x [k'_j + (\vec{\mu}\times\vec{B})_j](\partial_i B^2 dx^1 - \partial_i B^1 dx^2), \end{aligned} \quad (72)$$

in which $k'_j = m' v_j$, $p_l = m' v_l + (\vec{\mu}\times\vec{B})_l + \mathcal{O}(\theta)$ has been applied and we omit the second order terms in θ. The term $\delta\phi_{NCPS}$ represents the non-commutativity for both space and momentum. The first term in $\delta\phi_{NCPS}$ is a contribution purely from the non-commutativity of the momenta, the second term is a velocity dependent correction and the third term is a correction to the vortex of magnetic field. In 2 dimensional non-commutative plane, $\bar{\theta}_{ij} = \bar{\theta}\epsilon_{ij}$, and the two NC parameters θ and $\bar{\theta}$ are

related by $\bar{\theta} = 4\alpha^2(1-\alpha^2)/\theta$ [?]. When $\alpha = 1$, which will lead to $\bar{\theta}_{ij} = 0$, then the $\delta\phi_{NCPS}$ returns to $\delta\phi_{NCS}$, namely, the HMW phase on NC phase space will return to the HMW phase in NC space.

5, Noncommutative AC Effect for spin-1 partcles

I: AC effect for spin-1 particles on a commutative space time

In this section by following Ref.[?] we review briefly the Aharonov-Casher effect of a spin-1 particle on a commutative space time. The Lagrangian for a free spin-1 particle of mass m is

$$L = \bar{\phi}(i\beta^\nu \partial_\nu - m)\phi \, , \tag{73}$$

where the 10×10 matrices β_ν are generalization of the 4×4 Dirac gamma matrices, and it can be chosen as follows[?]-[?]

$$\beta^0 = \begin{pmatrix} \widehat{O} & \widehat{O} & I & o^\dagger \\ \widehat{O} & \widehat{O} & \widehat{O} & o^\dagger \\ I & \widehat{O} & \widehat{O} & o^\dagger \\ o & o & o & 0 \end{pmatrix} , \qquad \beta^j = \begin{pmatrix} \widehat{O} & \widehat{O} & \widehat{O} & -iK^{j\dagger} \\ \widehat{O} & \widehat{O} & S^j & o^\dagger \\ \widehat{O} & -S^j & \widehat{O} & o^\dagger \\ -iK^j & o & o & 0 \end{pmatrix} ,$$

with $j = 1, 2, 3$. The elements of the 10×10 matrices β_ν are given by the matrices

$$\widehat{O} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \qquad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} ,$$

$$S^1 = i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} , \quad S^2 = i\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} , \quad S^2 = i\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} ,$$

$$o = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}, \quad K^1 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \quad K^2 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}, \quad K^3 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} .$$

The above β matrices satisfy the following relation

$$\beta_\nu \beta_\lambda \beta_\rho + \beta_\rho \beta_\lambda \beta_\nu = \beta_\nu g_{\lambda\rho} + \beta_\rho g_{\nu\lambda}. \tag{74}$$

Other algebraic properties of the Kemmer β-matrices were given in Ref.[?]; the metric tensor is $g_{\lambda\rho} = \text{diag}(1, -1, -1, -1)$. The Kemmer equation of motion is

$$(i\beta^\nu \partial_\nu - m)\phi = 0. \tag{75}$$

The Lagrangian for a spin-1 neutral particle with a magnetic dipole moment μ_m interacting with the electromagnetic field has the form

$$L = \bar{\phi}(i\beta^\nu \partial_\nu + \frac{1}{2}\mu_m S_{\lambda\rho} F^{\lambda\rho} - m)\phi \ , \tag{76}$$

where $F^{\lambda\rho}$ is the field strength of the electromagnetic field; $S_{\lambda\rho}$ is the Dirac $\sigma_{\lambda\rho}$ like spin operator, which can be defined as

$$S_{\lambda\rho} = \frac{1}{2}(\beta_\lambda \beta_\rho - \beta_\rho \beta_\lambda) \ . \tag{77}$$

It follows that in the presence of an electromagnetic field, the Kemmer equation of motion of a spin-1 neutral particle with a magnetic moment μ_m is

$$(i\beta^\nu \partial_\nu + \frac{1}{2}\mu_m S_{\lambda\rho} F^{\lambda\rho} - m)\phi = 0 \ . \tag{78}$$

The aim is to find a solution of the above equation, which can be written in the following form

$$\phi = e^{-i\xi_3 \int^r \mathbf{A}' \cdot \, d\mathbf{r}} \ \phi_0, \tag{79}$$

where ϕ_0 is a solution of (75); the spin-1 pseudo-vector operator ξ_ν in (79) is defined as

$$\xi_\nu = \frac{1}{2} i \varepsilon_{\nu\lambda\rho\sigma} \beta^\lambda \beta^\rho \beta^\sigma \ , \tag{80}$$

where $\varepsilon_{\nu\lambda\rho\sigma}$ is the Levi-Civita symbol in four dimensions. Now we need to find the explicit form of the vector $\mathbf{A}'$ in (79). To do this, first we write Eq.(75) for ϕ_0 in terms of ϕ

$$(i\beta^\nu \partial_\nu - m) \ e^{i\xi_3 \int^r \mathbf{A}' \cdot \, d\mathbf{r}} \ \phi = 0. \tag{81}$$

Then the equivalence of (78) and (81) can be obtained by imposing the following two conditions

$$e^{-i\xi_3 \int^r \mathbf{A}' \cdot \, d\mathbf{r}} \ \beta^\nu \ e^{i\xi_3 \int^r \mathbf{A}' \cdot \, d\mathbf{r}} = \beta^\nu, \tag{82}$$

and

$$-\beta^{\nu}\xi_3 A'_{\nu}\phi = \frac{1}{2}\mu_m S_{\lambda\rho}F^{\lambda\rho}\phi = \mu_m S_{0l}F^{0l}\phi. \qquad (83)$$

By comparing Eq. (82) with the Baker-Housdorf formula

$$e^{-i\lambda\xi_3}\,\beta^{\nu}\,e^{i\lambda\xi_3} = \beta^{\nu} + \wp(-i\lambda)[\xi_3,\beta^{\nu}] + \frac{1}{2!}\wp(-i\lambda)^2[\xi_3,[\xi_3,\beta^{\nu}]]\ldots, \qquad (84)$$

one obtains, $[\xi_3,\beta^{\nu}] = 0$; $\wp$ in (84) stands for the path ordering of the integral in the phase. If $\nu \neq 3$ this commutation relation is automatically satisfied. For $\nu = 3$, by using (74) and (80), one finds that the commutator does not vanish. Therefore in order to fulfil the first condition the particle is restricted to move in $x-y$ plane, that is, $p_z = 0$. In particular $\partial_3\phi = 0$ and $\hat{A}'_3 = 0$. From second condition (83), by using (74), (77) and (80), one obtains

$$A'_1 = -2\mu_m E_2\,, \qquad A'_2 = 2\mu_m E_1\,. \qquad (85)$$

Thus the AC phase for a neutral spin-1 particle moving in a $2+1$ space time under the influence of a pure electric field produced by a uniformly charged infinitely long filament perpendicular to the plane is

$$\phi_{AC} = \xi_3\oint \mathbf{A}'\cdot\, d\mathbf{r} = 2\mu_m\xi_3\oint(E_1dx_2 - E_2dx_1) = 2\mu_m\xi_3\varepsilon^{lk}\oint E_l dx_k\,. \qquad (86)$$

The above equation can also be written as in Ref.[?]

$$\phi_{AC} = \xi_3\oint \mathbf{A}'\cdot\, d\mathbf{r} = \xi_3\int_S(\nabla\times\mathbf{A}')\cdot d\mathbf{S} = 2\mu_m\xi_3\int_S(\nabla\cdot\mathbf{E})dS = 2\mu_m\xi_3\lambda_e\,, \qquad (87)$$

where λ_e is the charge density of the filament. This spin-1 AC phase is a purely quantum mechanical effect and has no classical interpretation. One may note that the AC phase for spin-1 particles is exactly the same as in the case of spin-$\frac{1}{2}$, except that the spin and spinor have changed. The factor of two shows that the phase is twice that accumulated by a spin-$\frac{1}{2}$ particle with the same magnetic dipole moment coupling constant, in the same electric field.

II: AC effect for spin-1 particles on a non-commutative space

On a non-commutative space the coordinate and momentum operators satisfy the following commutation relations (we take $\hbar = c = 1$ unit)

$$[\hat{x}_i,\hat{x}_j] = i\Theta_{ij}, \quad [\hat{p}_i,\hat{p}_j] = 0, \quad [\hat{x}_i,\hat{p}_j] = i\delta_{ij}, \qquad (88)$$

where Θ_{ij} is an element of an antisymmetric matrix, it is related to the energy scale and it represents the non-commutativity of the NC space; $\hat{x}_i$ and $\hat{p}_i$ are the coordinate and momentum operators on a NC space.

By replacing the usual product in (78) with a star product (Moyal-Weyl product), the Kemmer equation for a spin-1 neutral particle with a magnetic dipole moment μ_m, on the NC space, can be written as

$$(i\beta^\nu \partial_\nu + \frac{1}{2}\mu_m S_{\lambda\rho} F^{\lambda\rho} - m) * \phi = 0, \tag{89}$$

The star product between two functions is defined by,

$$(f * g)(x) = e^{\frac{i}{2}\Theta_{ij}\partial_{x_i}\partial_{x_j}} f(x_i)g(x_j) = f(x)g(x) + \frac{i}{2}\Theta_{ij}\partial_i f \partial_j g|_{x_i=x_j} + \mathcal{O}(\Theta^2), \tag{90}$$

here $f(x)$ and $g(x)$ are two arbitrary functions.

On a NC space the star product can be changed into an ordinary product by a Bopp's shift, that is, by shifting coordinates x_ν with

$$x_\nu \longrightarrow \hat{x}_\nu = x_\nu - \frac{1}{2}\Theta_{\nu\lambda} p^\lambda. \tag{91}$$

Now, let us consider the non-commutative Kemmer equation (89). To replace the star product in (89) with an ordinary product, the $F_{\nu\lambda}$ must, up to the first order of the NC parameter Θ, be shifted [?] as

$$F_{\nu\lambda} \longrightarrow \hat{F}_{\nu\lambda} = F_{\nu\lambda} + \frac{1}{2}\Theta^{\rho\sigma} p_\rho \partial_\sigma F_{\nu\lambda}. \tag{92}$$

which is equivalent to the Bopp's shift (91). Then the Kemmer equation on a NC space has the form

$$(i\beta^\nu \partial_\nu + \frac{1}{2}\mu_m S_{\lambda\rho} \hat{F}^{\lambda\rho} - m)\phi = 0. \tag{93}$$

In a similar way as the commuting space, the solution of the above equation can also be written as

$$\phi = e^{-i\xi_3 \int^r \hat{\mathbf{A}}' \cdot\, d\mathbf{r}}\ \phi_0 \tag{94}$$

To determine $\hat{\mathbf{A}}'$ we write the free Kemmer equation as

$$(i\beta^\nu \partial_\nu - m)\ e^{i\xi_3 \int^r \hat{\mathbf{A}}' \cdot\, d\mathbf{r}}\ \phi = 0 \tag{95}$$

The equivalence of (93) and (95) gives the following two conditions

$$e^{-i\xi_3 \int^r \hat{\mathbf{A}}' \cdot d\mathbf{r}} \beta^\nu e^{i\xi_3 \int^r \hat{\mathbf{r}}' \cdot d\mathbf{r}} = \beta^\nu \tag{96}$$

and

$$-\beta^\nu \xi_3 \hat{A}'_\nu \phi = \frac{1}{2}\mu_m S_{\lambda\rho} \hat{F}^{\lambda\rho}\phi = \mu_m S_{0l} \hat{F}^{0l}\phi \tag{97}$$

By using the Baker-Housdorf formula (84), the first condition (96) implies that, $[\xi_3, \beta^\nu] = 0$. If $\nu \neq 3$ then this commutation relation is automatically satisfied. For $\nu = 3$, by using (74) and (80), one finds that the commutator does not vanish. Therefore in order to fulfil the first condition we restrict ourselves to $2+1$ space-time. In particular $\partial_3\phi = 0$ and $\hat{A}'_3 = 0$. From second condition (97), by using (74), (77) and (80), one obtains

$$\begin{aligned} \hat{A}'_1 &= -2\mu_m \hat{F}^{02} = -2\mu_m F^{02} - 2\mu_m \frac{1}{2}\Theta^{ij} p_i \partial_j F^{02} = -2\mu_m E_2 - \mu_m \theta \varepsilon^{ij} p_i \partial_j E_2 \\ \hat{A}'_2 &= 2\mu_m \hat{F}^{01} = 2\mu_m F^{01} + 2\mu_m \frac{1}{2}\Theta^{ij} p_i \partial_j F^{01} = 2\mu_m E_1 - \mu_m \theta \varepsilon^{ij} p_i \partial_j E_1 \end{aligned} \tag{98}$$

with $\Theta^{ij} = \theta\epsilon^{ij}$, $\Theta^{0\mu} = \Theta^{\mu 0} = 0$; $\epsilon^{ij} = -\epsilon^{ji}$, $\epsilon^{12} = +1$. Thus the AC phase for a neutral spin-1 particle moving in a $2+1$ non-commutative space under the influence of a pure electric field produced by a uniformly charged infinitely long filament perpendicular to the plane is

$$\hat{\phi}_{AC} = \xi_3 \oint \hat{\mathbf{A}}' \cdot d\mathbf{r} = 2\mu_m \xi_3 \varepsilon^{lk} \oint E_l dx_k + \mu_m \xi_3 \theta \varepsilon^{ij} \varepsilon^{lk} \oint p_i \partial_j E_l dx_k \,. \tag{99}$$

In a similar way as in spin-$\frac{1}{2}$ [?] [?], the momentum on a NC space for a spin-1 neutral particle can also be written as

$$p_i = mv_i + (\vec{E} \times \vec{\mu})_i + \mathcal{O}(\theta), \tag{100}$$

where $\vec{\mu} = 2\mu_m \vec{S}$, and $\vec{S}$ is the spin operator of the spin-1. By inserting (100) into (99), we have

$$\hat{\phi}_{AC} = \phi_{AC} + \delta\phi_{NCS}, \tag{101}$$

where ϕ_{AC} is the AC phase in (86) on a commuting space; the additional phase $\delta\phi_{NCS}$, related to the non-commutativity of space, is given by

$$\delta\phi_{NCS} = \mu_m \xi_3 \theta \varepsilon^{ij} \varepsilon^{lk} \oint [k_i - (\vec{\mu} \times \vec{E})_i] \partial_j E_l dx_k \tag{102}$$

where $k_i = mv_i$ is the wave number; ξ_3 represents spin degrees of freedom. If the spin of the neutral particle along the z direction, namely, $\vec{\mu} = 2\mu_m s_3 \hat{\vec{k}}$, $\hat{\vec{k}}$ is a unite vector in the z direction, then our results here are the same as the result of Ref. [?], where the star product calculation has been used.

III: AC effect for spin-1 particles on a non-commutative phase space

In section above we have investigated the AC effect for a neutral spin-1 particle on a NC space, where space-momentum, and space-space are non-commuting, but momentum-momentum are commuting. The Bose-Einstein statistics in non-commutative quantum mechanics requires both space-space and momentum-momentum non-commutating. The NC space with non-commuting momentum-momentum is called NC phase space. In this section we study the AC phase on a NC phase space. On a NC phase space, the commutation relation in (88) should be replaced by

$$[\hat{p}_i, \hat{p}_j] = i\bar{\Theta}_{ij}, \tag{103}$$

where $\bar{\Theta}$ is the antisymmetric matrix, its elements represent the non-commutative property of the momenta. Then the Kemmer equation for AC problem on a NC phase space has the form

$$(-\beta^\nu p_\nu + \frac{1}{2}\mu_m S_{\lambda\rho} F^{\lambda\rho} - m) * \phi = 0. \tag{104}$$

The star product in (104) on a NC phase space can be replaced by the usual product in two steps, first we need to replace x_i and p_i by a generalized Bopp's shift as

$$\begin{aligned} x_\nu \rightarrow \hat{x}_\nu &= \alpha x_\nu - \frac{1}{2\alpha}\Theta_{\nu\lambda} p^\lambda, \\ p_\nu \rightarrow \hat{p}_\nu &= \alpha p_\nu + \frac{1}{2\alpha}\bar{\Theta}_{\nu\lambda} x^\lambda, \end{aligned} \tag{105}$$

where α is the scaling parameter, and it is related to the non-commutativity of the phase space via $\Theta\bar{\Theta} = 4\alpha^2(\alpha^2 - 1)\cdot \mathbf{I}$, here $\mathbf{I}$ is a unit matrix. Then we also need to rewrite the shift in (92) as

$$F_{\nu\lambda} \rightarrow \hat{\mathcal{F}}_{\nu\lambda} = \alpha F_{\nu\lambda} + \frac{1}{2\alpha}\Theta^{\rho\sigma} p_\rho \partial_\sigma F_{\nu\lambda}. \tag{106}$$

Thus the Kemmer equation for AC problem on a NC phase space has the form

$$(-\beta^{\nu}\hat{p}_{\nu}+\frac{1}{2}\mu_m S_{\lambda\rho}\hat{\mathcal{F}}^{\lambda\rho}-m)\phi=0. \quad (107)$$

Since $\alpha\neq 0$, the above equation can be written as

$$\big(-\beta^{\nu}p_{\nu}-\frac{1}{2\alpha^2}\beta^{\nu}\bar{\Theta}_{\nu\lambda}x^{\lambda}+\frac{1}{2}\mu_m S_{\lambda\rho}(F^{\lambda\rho}+\frac{1}{2\alpha^2}\Theta^{\sigma\tau}p_{\sigma}\partial_{\tau}F^{\lambda\rho})-m')\phi=0. \quad (108)$$

where $m'=m/\alpha$. We write the above equation in the following form

$$(-\beta^{\nu}p_{\nu}-m')\ e^{\frac{i}{2\alpha^2}\int^r\bar{\Theta}_{\nu\lambda}x^{\lambda}dx^{\nu}+i\xi_3\int^r\hat{\mathcal{A}}'\cdot\,d\mathbf{r}}\ \phi=0\ . \quad (109)$$

To have the equivalence of (108) and (109), we impose the following two conditions

$$e^{-i\xi_3\int^r\hat{\mathcal{A}}'\cdot\,d\mathbf{r}}\ \beta^{\nu}\ e^{i\xi_3\int^r\hat{\mathcal{A}}'\cdot\,d\mathbf{r}}=\beta^{\nu}\ , \quad (110)$$

and

$$-\beta^{\nu}\xi_3\hat{\mathcal{A}}'_{\nu}\phi=\frac{1}{2\alpha}\mu_m S_{\lambda\rho}\hat{F}^{\lambda\rho}\phi=\frac{\mu_m}{\alpha}S_{0l}\hat{F}^{0l}\phi\ . \quad (111)$$

In an analogous way as in NC space, from (110) and (111) one may obtain

$$\begin{aligned}
\hat{\mathcal{A}}'_1 &= -\frac{2\mu_m}{\alpha}\hat{F}^{02}=-2\mu_m F^{02}-2\mu_m\frac{1}{2\alpha^2}\Theta^{ij}p_i\partial_j F^{02}=-2\mu_m E_2-\frac{\mu_m\theta}{\alpha^2}\varepsilon^{ij}p_i\partial_j E_2\ ,\\
\hat{\mathcal{A}}'_2 &= \frac{2\mu_m}{\alpha}\hat{F}^{01}=2\mu_m F^{01}+2\mu_m\frac{1}{2\alpha^2}\Theta^{ij}p_i\partial_j F^{01}=2\mu_m E_1+\frac{\mu_m\theta}{\alpha^2}\theta\varepsilon^{ij}p_i\partial_j E_1\ ,\\
\hat{\mathcal{A}}'_3 &= 0\ .
\end{aligned} \quad (112)$$

Thus the AC phase for a neutral spin-1 particle moving in a $2+1$ noncommutative phase space under the influence of a pure electric field produced by a uniformly charged infinitely long filament perpendicular to the plane is given by

$$\begin{aligned}
\hat{\phi}_{AC} &= \frac{1}{2\alpha^2}\oint\bar{\Theta}_{\nu\lambda}x^{\lambda}dx^{\nu}+\xi_3\oint\hat{\mathcal{A}}'\cdot\,d\mathbf{r}\\
&= \frac{\theta}{2\alpha^2}\oint\varepsilon^{ij}x_j dx_i+2\mu_m\xi_3\varepsilon^{lk}\oint E_l dx_k+\mu_m\xi_3\frac{\theta}{\alpha^2}\varepsilon^{ij}\varepsilon^{lk}\oint p_i\partial_j E_l dx_k
\end{aligned} \quad (113)$$

By $p_i = k'_i + (\vec{E} \times \vec{\mu})_i + \mathcal{O}(\theta)$, and $k'_i = m'_i v_i$, $\vec{\mu} = 2\mu_m \vec{S}$, one obatains

$$\hat{\varphi}_{AC} = \phi_{AC} + \delta\phi_{NCS} + \delta\phi_{NCPS}, \tag{114}$$

where ϕ_{AC} is the AC phase in (86) on a commuting space; $\delta\phi_{NCS}$ is the space-space non-commuting contribution to the AC phase in (86) , and its explicit form is given in (102); the last term $\delta\phi_{NCPS}$ is the momentum-momentum non-commuting contribution to the AC phase in (86), and it has the form

$$\delta\phi_{NCPS} = \frac{\bar{\theta}}{2\alpha^2} \oint \varepsilon^{ij} x_j dx_i + (\frac{1}{\alpha^2} - 1)\mu_m \xi_3 \theta \varepsilon^{ij} \varepsilon^{lk} \oint [k'_i - (\vec{\mu} \times \vec{E})_i] \partial_j E_l dx_k \tag{115}$$

which represents the non-commutativity of the momenta. The first term in (115) comes from the momentum-momentum non-commutativity; the second term is a velocity dependent correction and does not have the topological properties of the commutative AC effect and could modify the phase shift. the third term is a correction to the vortex and does not contribute to the line spectrum. In 2-dimensional non-commutative plane, $\bar{\Theta}_{ij} = \bar{\theta}\epsilon_{ij}$, and the two NC parameters θ and $\bar{\theta}$ are related by $\bar{\theta} = 4\alpha^2(1-\alpha^2)/\theta$ [?]. When $\alpha = 1$, which leads to $\bar{\theta}_{ij} = 0$, the AC phase on a NC phase space case reduces to the AC phase on a NC space case, i.e. $\delta\phi_{NCPS} = 0$ and equation (114) changes into equation (101).

6, Conclusions.

The main results in my talk is given as follows.

(1) **Representation of $\hat{x}_i, \hat{p}_i$ (of NC phase space) in terms of commutative coordinates and momenta:**

$$\begin{cases} \hat{x}_i &= \alpha x_i - \frac{1}{2\alpha}\Theta_{ij}p_j, \\ \hat{p}_i &= \alpha p_i + \frac{1}{2\alpha}\bar{\Theta}_{ij}x_j. \end{cases} \tag{116}$$

(2) **Schr$\ddot{o}$dinger equation:**

On NC space

$$H(\hat{x}_i, \hat{p}_i)\psi = H(x_i - \frac{1}{2\hbar}\Theta_{ij}p_j, p_i)\psi = E\psi. \tag{117}$$

On NC phase space,

$$H(x_i - \frac{1}{2\hbar\alpha^2}\Theta_{ij}p_j, p_i + \frac{1}{2\hbar\alpha^2}\bar{\Theta}_{ij}x_j)\psi = E\psi. \tag{118}$$

(3) **The AC effect**

On NC Phase space for spin-1/2

$$\delta\phi_{NCPS} = \frac{1}{2\alpha^2}\int^x \bar{\Theta}_{ij}x_j dx_i + \frac{1}{2\alpha^2}\gamma^0\sigma^{12}\mu_m\theta\epsilon^{ij}\int^x [k'_j + (\vec{E}\times\vec{\mu})_j](\partial_i E^2 dx^1 - \partial_i E^1 dx^2)$$

On NC Phase space for spin-1

$$\delta\phi_{NCPS} = \frac{\bar{\theta}}{2\alpha^2}\oint \varepsilon^{ij}x_j dx_i + (\frac{1}{\alpha^2} - 1)\mu_m\xi_3\theta\varepsilon^{ij}\varepsilon^{lk}\oint [k'_i - (\vec{\mu}\times\vec{E})_i]\partial_j E_l dx_k \tag{120}$$

(4) **The HMW effect,**

On NC Phase space,for spin-1/2

$$\hat{\varphi}_{HMW} = \phi_{HMW} + \delta\phi_{NCPS}, \tag{121}$$

where

$$\delta\phi_{NCPS} = -\frac{1}{2\alpha^2}\int^x \bar{\theta}_{ij}x_j dx_i - \frac{1}{2\alpha^2}\mathbf{a}\mu_e \int^x \epsilon_{\mu 0 i}\theta\epsilon^{\alpha\beta}[m' v_\alpha + (\vec{\mu}\times\vec{B})_\alpha]\partial_\beta \tilde{F}^{0i} dx^\mu$$
$$= -\frac{1}{2\alpha^2}\int^x \bar{\theta}_{ij}x_j dx_i + \frac{1}{2\alpha^2}\mathbf{a}\mu_e\theta\epsilon^{ij}\int^x [k'_j + (\vec{\mu}\times\vec{B})_j](\partial_i B^2 dx^1 - \partial_i B^1 dx^2), (1$$

On NC Phase space for spin-1

$$\begin{aligned}\delta\phi_{NCPS} &= \frac{1}{2\alpha^2}\oint \bar{\Theta}_{\nu\lambda}x^\lambda dx^\nu + \xi_3\oint \hat{\mathbf{a}}' \cdot\ d\mathbf{r} \\ &= \frac{\theta}{2\alpha^2}\oint \varepsilon^{ij}x_j dx_i + 2\mu_e\xi_3\varepsilon^{lk}\oint B_l dx_k \\ &\quad +\mu_e\xi_3\frac{\theta}{\alpha^2}\varepsilon^{ij}\varepsilon^{lk}\oint (k'_i + (\vec{B}\times\vec{\mu})_i)\partial_j B_l dx_k \end{aligned} \tag{123}$$

THANKS!!!