PERTURBATION EXPANSIONS IN QFT

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Abstract

Perturbation theory is a powerful tool in manipulating dynamical system. However, it is legal only for infinitesimal perturbations. We propose to dispose this problem by means of perturbation group, and find that the coupling constant approaches to zero in the limit of high order perturbations as Dyson once expected.

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1 Motivations

Perturbation theory + renormalization schemes:$\Rightarrow$ QM, QFT. QED, prominent success in predictions for

1. Lamb shift: $\Delta\nu \equiv (E_{2s} - E_{2p})/2\pi\hbar$,

$$\begin{array}{rcl} \Delta\nu_{QED} & = & 1420.45199(14)MHz, \\ \Delta\nu_{\exp} & = & 1420.4057517667(9)MHz, \end{array}$$

2. Anormalous magnetic moment g of electron in units of Bohr magneton $\mu_B = e\hbar/2mc$. $a \equiv (g-2)/2$,

$$\begin{array}{rcl} a_{\rm QED} & = & 0.001159652174.19(\text{from Kinoshita}) \\ a_{\rm exp} & = & 0.0011591628(77) \\ a_{\rm QED}^{\rm PG} & = & 0.00116052601 \end{array}$$

Dyson: All the asymptotic series used in quantum electrodynamics after renormalization in mass and charge are divergent.

Jaffe: Cut from the origin to $-\infty$ along the real axis in the complex plane of g.

Bender and Wu : 1D- ϕ^4-model $\Rightarrow$ 1. Analytic behaviors for the eigenfunction $\phi(x, \lambda)$, eigenvalue $E(\lambda)$ and resolvent $(z - H)^{-1}$. 2. The asymptotic series for E_0 is divergent. 3. An infinite sequence of poles for the resolvent when the phase of $g \to \pm\frac{3}{2}\pi$.

Lipatov: the renormalizable polynomial interaction scalar model$\Rightarrow$1. The dominated contributions come from those diagrams with certain numbers of vertices connected by equal numbers of internal lines. 2. Watson-Sommerfeld transformation$\mapsto$ the existence of the ultraviolet fixed point of the theory.

Brezin et. al.: Anharmonic oscillators$\Rightarrow$Generalized the result to the case of internal $O(n)$ symmetry.

Kazakov and Popov:Asymptotic of the Gell-Mann-Low β-function cannot be recovered by its first coefficients of the perturbation series and their asymptotic values without invoking additional information.

Mathematical works:

Kato: Theorem for the analytic behavior of of the eigenvalues of operators in analytic family.

Reed and Simon: Systematic and extensive review for the theory of perturbation for bounded as well as unbounded operators.

2 Perturbation expansions in QED

Keiichi Ito:

Therefore the present study will cast a light on new and constructive study of 4D electrodynamics which is believed trivial by most of modern physicists. (Trivial means **that the full theory of QED does not exist and the perturbation expansion has nothing to do with the physical phenomena in the world of electrons and photons.**)

The application of perturbation expansions in QED achieved great success in QED. This was first explored by Karplus and Kroll where the covariant S matrix formalism of Dyson was applied to the calculation of the fourth-order radiative correction to the magnetic moment of the electron. Then Kinosita went further to 10th order. Among them the key step was that taken by Karplus and Kroll in Harvard in 1954. Ins'nt there any problem? May be.

1. The first Born approximation was made by Karplus and Kroll in their manipulation. But according to Dyson one may go further to higher Born approximation

$$U = \sum_{n,m=0}^{\infty} \left(\frac{-i}{\hbar c}\right)^{n+m} \frac{1}{n!m!} \int_{-\infty}^{\infty} d^4x_0 d^4x_1 ... d^4x_{n+m} P[H^e(x_0), H^e(x_0), ..., H^I(x_1)...H^I(x_n)], \quad (1)$$

2. It is likely that there might be something worthy further inspection.

Our propositions: 1. Perturbation group, 2. Coupling constant $g(n) \to 0$.

3 Perturbation group

Asymptotic series for $f(\lambda)$:

$$\lim_{\lambda \downarrow 0} \Big(f(\lambda) - \sum_{n=0}^{N} a_n \lambda^n\Big) \Big/ \lambda^N = 0 \quad (2)$$

If the asymptotic series is not convergent, then its typical behavior is that the first few partial sums are fairly good approximation, but in higher approximations the sums oscillate widely and no longer approximate the desired limit.

Example 1: $f(z) = \exp(-z^{-1})$ for $z > 0$.

$\lim_{z \downarrow 0} z^{-n}[f(z) - \sum_n 0 \cdot z^n] \to 0$, $\Rightarrow f$ has 0 as its asymptotic series. In fact , the zero series is asymptotic uniformly in any sector $|\arg z| \leq \theta$ with $\theta < \pi/2$.

Two different functions may have the same asymptotic series. Saying that f has a certain asymptotic series gives us no information about the values of $f(z)$ for some fixed nonzero value of z.

Example 2: Rayleigh-Schrödinger Series for Ground State Energy:

The ground state energy of the Hamiltonian, $p^2 + x^2 + \beta x^4 (\beta > 0)$, is the asymptotic to $E_0(\beta)$ as $\beta \downarrow 0$. For $\beta = 0.2$, variational methods show that $E_0(\beta) = 1.118292.....$ The first 15 partial sums are given in the following table.

N	$\sum_{n=0}^{N} a_n(0.2)^n$	N	$\sum_{n=0}^{N} a_n(0.2)^n$
1	1.150000	9	2.353090
2	1.097500	10	-2.442698
3	1.153750	11	13.253698
4	1.105372	12	-42.333586
5	1.176999	13	168.895730
6	1.049024	14	-796.466406
7	1.413970	15	3005.179546
8	0.686006	50	$\sim 10^{45}$

Thus we see the typical behavior of wandering near the right answer for a while (and not even that near!) and then going wild. And as N gets larger, things get worse.

Our proposition: Separate the perturbation into many steps and limit the perturbation to be infinitesimal in each step.$\Rightarrow$ Perturbation group

Let $\beta \in \mathbb{Z}, H, K \in \mathcal{B}$(Banach space—Linear normed space), then

$$H = H_0 + \beta K \in \mathcal{B}.$$

Let $T(\beta, \beta_0)$ be a translation in $\mathcal{B}$,

$$T(\beta, \beta_0): \qquad H_{\beta_0} = H_0 + \beta_0 K \longmapsto H_\beta = H_0 + \beta K. \tag{3}$$

Let the representation $U(\beta, \beta_0)$ of the translation $T(\beta, \beta_0)$ be a transformation on a Hilbert space $\mathcal{H}$ defined by $U(\beta_1, \beta_0) \in \mathcal{L}(\mathcal{H})$,

$$U(\beta_1, \beta_0)\varphi_{\beta_0} = \varphi_{\beta_1} \tag{4}$$

where φ_{β_0} and φ_{β_1} are the eigenvectors of H_{β_0} and H_{β_1} respectively,

$$\begin{aligned} H_{\beta_0}\varphi_{\beta_0} \equiv &\quad (H_0 + \beta_0 K)\varphi_{\beta_0} = \lambda_{\beta_0}\varphi_{\beta_0}, \\ H_{\beta_1}\varphi_{\beta_1} \equiv &\quad (H_0 + \beta_1 K)\varphi_{\beta_0} = (H_{\beta_0} + (\beta_1 - \beta_0)K)\varphi_{\beta_1} = \lambda_{\beta_1}\varphi_{\beta_1}, \end{aligned} \tag{5}$$

and it is assumed that when β varies continuously from β_0 to β_1, the eigenvector of $H(\beta, \beta_0))$ varies continuously from $\varphi(\beta_0)$ to $\varphi(\beta_1)$. Since

$$U(\beta_2, \beta_1)U(\beta_1, \beta_0)\varphi_{\beta_0} = U(\beta_2, \beta_1)\varphi_{\beta_1} = \varphi_{\beta_2}, \tag{6}$$

and

$$U(\beta_2, \beta_0)\varphi_{\beta_0} = \varphi_{\beta_2}, \tag{7}$$

Thus

$$U(\beta_2, \beta_1)U(\beta_1, \beta_0) = U(\beta_2, \beta_0). \tag{8}$$

Besides

$$U(\beta, \beta) = 1, \tag{9}$$

and

$$U(\beta_1, \beta_0)^{-1} = U(\beta_0, \beta_1). \tag{10}$$

Therefore the transformations $U(\beta_1, \beta_0)$ form a group.

4 Generator for perturbation group

Let us first setup the perturbation equation for the perturbation group. The derivative of $U(\beta, \beta_0)$ with respect to β is given as follows,

$$\begin{aligned} \frac{\partial U(\beta, \beta_0)}{\partial \beta} &= \lim_{\Delta\beta \to 0} \frac{U(\beta + \Delta\beta, \beta_0) - U(\beta, \beta_0)}{\Delta\beta} \\ &= \lim_{\Delta\beta \to 0} \frac{U(\beta + \Delta\beta, \beta) - U(\beta, \beta)}{\Delta\beta} U(\beta, \beta_0) \\ &= \left.\frac{\partial U(\beta', \beta)}{\partial \beta'}\right|_{\beta' = \beta} U(\beta, \beta_0). \end{aligned} \tag{11}$$

where we denote

$$G(\beta) \equiv \left.\frac{\partial U(\beta', \beta)}{\partial \beta'}\right|_{\beta' = \beta} \tag{12}$$

as the generator of the perturbation group. thus we have the perturbation equation for the perturbation group as follows,

$$\frac{\partial U(\beta, \beta_0)}{\partial \beta} = G(\beta)U(\beta, \beta_0). \tag{13}$$

The group is hermitian if and only if $G(\beta)$ is anti-hermitian,

$$G^{\dagger}(\beta) = -G(\beta). \tag{14}$$

The infinitesimal transformation can be written as

$$U(\beta + \Delta\beta, \beta) = \exp\{G(\beta)\Delta\beta\}, \quad \text{for} \quad \Delta\beta \to 0 \tag{15}$$

Therefore we can proceed with the infinitesimal perturbations step by step, and finally obtain the perturbation with finite coupling constant. Let C be a section of line connecting two points β_0 and β_1 in the analytic regime of the complex plane of β, while the line segment is divided into n sections by $n-1$ points $\beta(s_k), s_k \in \mathbb{R}, (k = 1, 2, 3, ..., n-1)$ on the line,

$$\beta(s_k) = \beta(s_0) + k\Delta\beta, \quad (k \in \mathbb{I}) \quad \beta(s_n) = \beta(s_f). \tag{16}$$

$$\begin{aligned}
U(\beta_f, \beta_0) = & \lim_{n\to\infty} U(\beta_0 + n\Delta\beta, \beta_0 + (n-1)\Delta\beta)....U(\beta_0 + k\Delta\beta, \beta_0 + (k-1)\beta))...U(\beta_0 + \Delta\beta, \beta_0) \\
= & \lim_{n\to\infty} \mathbb{P} \exp\{G(\beta_0 + (n-1)\Delta\beta)\Delta\beta\}... \exp\{G(\beta_0)\Delta\beta\} \\
= & \mathbb{P} \exp\{\lim_{n\to\infty} \sum_{k=0}^{n-1} G(\beta + k\Delta\beta)\Delta\beta)\} \\
= & \mathbb{P} \exp\left\{ \int_{\beta_0}^{\beta_f} G(\beta) d\beta \right\} \\
= & 1 + \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\beta_0}^{\beta_f} d\beta_1 \int_{\beta_0}^{\beta_f} d\beta_2 ... \int_{\beta_0}^{\beta_f} d\beta_n \mathbb{P} G(\beta_1) G(\beta_2) ... G(\beta_n).
\end{aligned} \tag{17}$$

where $\mathbb{P}$ denotes the perturbation ordered product by the perturbation constant $\beta \in \mathbb{R}$,

$$\mathbb{P} G(\beta(s_2)) G(\beta(s_1)) = \begin{cases} G(\beta(s_2)) G(\beta(s_1)), & \text{for} \quad s_2 > s_1; \\ G(\beta(s_1)) G(\beta(s_2)), & \text{for} \quad s_1 > s_2, \end{cases} \tag{18}$$

and use is made of the fact that

$$\mathbb{P} \int_{\beta_0}^{\beta_f} d\beta_1 \int_{\beta_0}^{\beta_1} d\beta_2 ... \int_{\beta_0}^{\beta_{n-1}} d\beta_n G(\beta_1) G(\beta_2) ... G(\beta_n) = \frac{1}{n!} \int_{\beta_0}^{\beta_f} d\beta_1 \int_{\beta_0}^{\beta_f} d\beta_2 ... \int_{\beta_0}^{\beta_f} d\beta_n \mathbb{P} G(\beta_1) G(\beta_2) ... G(\beta_n). \tag{19}$$

Note that in view of the of the perturbation ordered product, there are no commutators between the generators in the exponential in contrast to the Campbell-Baker-Hausdorff formula.

The transformation $U(\beta_f, \beta_0)$ in eq.(17) can be manipulated through iteration, i. e., first insert the zeroth approximation into the right hand side integrals and obtain the first approximation, then iterate.

5 Perturbation group for quantum field theories

The evolution of the state vector $\varphi_I(\beta; t)$ in the interaction picture is effected by the evolution operator $U(\beta, \beta_0; t_1, t_0)$,

$$i \frac{\partial U(\beta_0 + \Delta\beta, \beta_0; t, t_0)}{\partial t} = \Delta\beta H_I(t) U(\beta_0 + \Delta\beta, \beta_0; t, t_0), \tag{20}$$

where $H_I(t) \equiv \int d^3x \mathcal{H}(t, \mathbf{x})$ is the Hamiltonian 3-density, and

$$U(\beta_0 + \Delta\beta, \beta_0; t, t_0) = \mathbb{T} \exp\{i \int dt \Delta\beta H_I(t)\} \tag{21}$$

Thus the generator for the infinitesimal perturbation transformation is

$$G(\beta) = i \int dt H_I(t), \tag{22}$$

Then the perturbation group equation is

$$\frac{\partial U(\beta_1, \beta_0; t, t_0)}{\partial \beta} = \Big\{ \int dt H_I(t) \Big\} U(\beta_1, \beta_0; t, t_0) \tag{23}$$

The perturbation transformation $U(\beta, 0; t, t_0)$ is

$$\begin{aligned} U(\beta, 0; t, t_0) = & \ \mathbb{P}\mathbb{T} \exp \Big\{ -i \int_0^\beta d\beta' \int_{t_0}^t dt H_I(t') \Big\} \\ = & \ 1 + \sum_n \frac{1}{n!} \frac{(-i)^n}{n!} \beta^n \int_{t_0}^t dt'_1 ... \int_{t_0}^t dt'_n \mathbb{T} H_I(t'_1) ... H_I(t'_n). \end{aligned} \tag{24}$$

In view of the reducing factor $\frac{1}{n!}$ before the n-th order term in the series, it seems that the contributions of the higher corrections are overestimated in the conventional series.[9]. This reducing factor $\frac{1}{n!}$ can also be absorbed into the coupling constant, thus the coupling constant is defined as a function $g(n)$ of order of approximation,

$$g(n) \equiv \frac{g}{(n!)^{1/n}}. \tag{25}$$

then the perturbation series recover its ordinary form. Using Stirling's formula, it is easily seen that

$$g(n) = \frac{g}{(n!)^{1/n}} \approx \frac{e}{n} g, \tag{26}$$

for large n, where e is the constant of the natural logarithm. Therefore

$$\lim_{n \to \infty} g(n) = 0, \tag{27}$$

as Dyson once expected[3].

6 Quantum electrodynamics and quantum chronodynamics

It is well-known that quantum electrodynamics achieved great success in theoretical prediction for anomalous magnetic moment of electron[15]. In quantum electrodynamics, according to Furry's theorem, the Feynman diagrams containing an odd number of photon vertices lead to vanishing contribution. Thus it seems to be more reasonable to choose the coupling constant $\alpha \equiv e^2$ as the parameter in loop expansions. Then we have the renormalized coupling constant α as

$$\alpha = Z_3 \alpha_B, \tag{28}$$

where α is the renormalized coupling constant, while α_B is the bare one.

The Landè factor g for anomalous magnetic moment of pure quantum electrodynamics [15, 16] is

$$g = 2 \Big[1 + C_1 \Big(\frac{\alpha}{\pi}\Big) + C_2 \frac{1}{2!} \Big(\frac{\alpha}{\pi}\Big)^2 + C_3 \frac{1}{3!} \Big(\frac{\alpha}{\pi}\Big)^3 + C_4 \frac{1}{4!} \Big(\frac{\alpha}{\pi}\Big)^4 + ... \Big], \tag{29}$$

where the coefficients C_i's are obtained in renormalization.

Therefore the g-factor from perturbation group approach is

$$g_{\rm QED}^{\rm PG} = 2(1 + 0.001160526044..), \tag{30}$$

while recent experimental data is [15]

$$g_{\rm exp} = 2(1 + 0.0011596521884...). \tag{31}$$

and ordinary perturbation data gives

$$g_{\rm QED}^{\rm Ord} \quad = \quad 2(1 + 0.0011596521564...) \tag{32}$$

Besides, one might still worry about the higher term may spoil the convergence of the asymptotic series in ordinary manipulations [3], since one cannot refuse to add the higher corrections which will eventually lead to divergence of the asymptotic series, while the situation in the perturbation group scheme is substantially improved. In going to higher corrections, the corrections decrease 10^{-3} with the order before the eighth order, but remains in the same order at the tenth order. This might be a signal of divergency.

In quantum chronodynamics, the Gell-Mann-Low β-function in the perturbation scheme can be defined by the following series [17],

$$\beta(g) = -\beta_0 g^3 - \beta_1 \frac{1}{2!} g^5 - \beta_2 \frac{1}{3!} g^7 + O(g^9), \tag{33}$$

where the coefficient can be obtained from renormalization,

$$\begin{aligned} \beta_0 = &\quad \frac{1}{(4\pi)^2}\Big(11 - \frac{2}{3}N_f\Big), \\ \beta_1 = &\quad \frac{1}{(4\pi)^4}\Big(102 - \frac{38}{3}N_f\Big), \\ \beta_2 = &\quad \frac{1}{(4\pi)^6}\Big(\frac{2857}{54} - \frac{5033}{18}N_f + \frac{325}{54}N_f^2\Big) \end{aligned} \tag{34}$$

where N_f is the number of flavors of quarks. Thus it is evident that the behavior of the ordinary β-function is substantially modified by the factors $1/n!$ in the perturbation group scheme.

7 Rayleigh-Schrödinger Series for Ground State Energy

The ground state energy of the Hamiltonian, $p^2 + x^2 + \beta x^4 (\beta > 0)$, is the asymptotic to $E_0(\beta)$ as $\beta \downarrow 0$.

In perturbation group scheme, for $\beta = 0.2$, the sums are given in the following table

N	$a_n(0.2)^n/n!$	N	$a_n(0.2)^n/n!$
0+1	+1.150000	9	+0.000005
2	-0.026250	10	-0.000001
3	+0.009375	11	+.0000000
4	-0.002016	12	-0.000000
5	+0.000597	13	+0.000000
6	-0.000178	14	-0.000000
7	+0.000052	15	+0.000000
8	-0.000016	$\sum$	1.131568

$E_0 \approx 1.131568$, rather than the data $E_0(\beta) = 1.118292....$ given by the variational method.

Actually, the Hamiltonian of the one-dimensional anharmonic oscillator is

$$H = \frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}m^2\varphi^2 + \lambda\varphi^4. \tag{35}$$

Then the ground state energy of the oscillator as observed by Vender and Wu is given by

$$E_0(\lambda) = \frac{1}{2}m + \sum_{n=1}^{\infty} mA_n(\lambda/m^3)^n. \tag{36}$$

The detail asymptotic growth of A_n is

$$A_n \sim (-1)^{n+1}(6/\pi^3)^{1/2}\Gamma(n+\frac{1}{2})3^n. \tag{37}$$

where

$$\Gamma\big(n+\frac{1}{2}\big) = \frac{(2n)!\sqrt{\pi}}{4^n n!} \tag{38}$$

According to Stirling's formula

$$\Gamma(x+1) = \sqrt{2\pi x}x^x e^{-x}(1+O(x^{\frac{1}{5}})) \tag{39}$$

Thus in the perturbation group scheme,

$$\begin{aligned} A_n \sim\ & (-1)^{n+1}\Big(\frac{6}{\pi^3}\Big)^{1/2}\frac{\Gamma\big(n+\frac{1}{2}\big)3^n}{n!} \\ =\ & (-1)^{n+1}\Big(\frac{6}{\pi^3}\Big)^{1/2}\frac{(2n)!\sqrt{\pi}3^n}{4^n(n!)^2} \\ =\ & (-1)^{n+1}\Big(\frac{6}{\pi^3}\Big)^{1/2}\sqrt{\pi}\frac{\sqrt{2\pi(2n+1)}}{2\pi(n+1)}\frac{(2n+1)^{2n+1}}{(n+1)^{2(n+1)}}\frac{e^{-(2n+1)}}{e^{-2(n+1)}}\Big(\frac{3}{4}\Big)^n \\ \to\ & (-1)^{n+1}\Big(\frac{6}{\pi^3}\Big)^{1/2}\frac{2^{2n+1}e}{(n+1)^{2/3}}\Big(\frac{3}{4}\Big)^n \\ \to\ & (-1)^{n+1}\Big(\frac{6}{\pi^3}\Big)^{1/2}\frac{2e3^n}{(n+1)^{3/2}} \end{aligned} \tag{40}$$

It can be easily seen that **if set $m=1$, and $\lambda < \frac{1}{3}$, then E_0 will go to a finite value in the limit of $n \to \infty$.**

8 Conclusion

We propose the perturbation group and perturbation equation, and then obtain the formalism for the perturbation transformation. Using the Rayleigh-Schrodinger series, we derive the generator for perturbation transformations, give out the transformation for finite perturbations, and find that the coupling constant in quantum field theory is varied and goes to zero as the order of approximation goes to infinity in ordinary perturbation series as Dyson once expected.

Acknowledgement The author would like express his hearty thanks to the Center of Cross Disciplines in Science sponsored by Professor L. Yu and the Institute for Theoretical Physics of the Academy of Science of China for their hospitality during the completion of part of this work. The author would also like to thank Professors H.-Y. Guo, K. Wu and W.-Z. Zhao for helpful discussions. This work is supported by NSFC no. 10265003.

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